# DUALIZING COMPLEX OF A TORIC FACE RING II: NON-NORMAL CASE

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Abstract. The notion of *toric face rings* generalizes both Stanley-Reisner rings and affine semigroup rings, and has been studied by Bruns, Römer, et.al. Here, we will show that, for a toric face ring $R$, the "graded" Matlis dual of a Cěch complex gives a dualizing complex. In the most general setting, $R$ is not a graded ring in the usual sense. Hence technical argument is required.

## 1. Introduction

Stanley-Reisner rings and affine semigroup rings are important subjects of combinatorial commutative algebra. The notion of *toric face rings*, which originated in an earlier work of Stanley [8], generalizes both of them, and has been studied by Bruns, Römer, and their coauthors recently (e.g. [1, 2, 4]). Contrary to these classical examples, a toric face ring does not admit a nice multi-grading in its most general setting. To make a toric face ring $R$ from a finite regular cell complex $\mathcal{X}$, we assign each cell $\sigma \in \mathcal{X}$ an affine semigroup $\mathbf{M}_\sigma \subset \mathbb{Z}^{\dim \sigma + 1}$ with $\mathbb{Z}\mathbf{M}_\sigma = \mathbb{Z}^{\dim \sigma + 1}$ so that certain compatibility is satisfied, and "glue" the affine semigroups $\Bbbk[\mathbf{M}_\sigma]$ of $\mathbf{M}_\sigma$ along with $\mathcal{X}$. (Note that not all $\mathcal{X}$ can support toric face rings.)

In the previous paper ([6]), Okazaki and the author gave a concise description of a dualizing complex of $R$ under the assumption that $\Bbbk[\mathbf{M}_\sigma]$ is normal for all $\sigma \in \mathcal{X}$. In the present paper, we treat the general (i.e., non-normal) case. While the result in [6] does not hold verbatim, we can show that the "Matlis dual" $(L_R^\bullet)^\vee$ of the Cěch complex $L_R^\bullet$ associated with the cell complex $\mathcal{X}$ is quasi-isomorphic to the dualizing complex. If $R$ itself is an affine-semigroup ring, $(L_R^\bullet)^\vee$ is the multigraded dualizing complex given by [5]. More generally, if $R$ has a nice-multigrading, this fact was already proved by Ichim and Römer [4]. In their case, standard argument using the graded ring structure works, but the general case requires much more technical argument. It would be an interesting problem to find another class of rings whose dualizing complexes are given by a similar way.

## 2. Notation and Preliminaries

In this section, we recall the construction and basic properties of a toric face ring. See [2, 6] for detail. We basically use the convention of [6].

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Let $\mathcal{X}$ be a finite regular cell complex with the intersection property, and $X$ its underlying topological space. More precisely, $\mathcal{X}$ is a finite set of subsets (called *cells*) of $X$ satisfying the following conditions.

(1) $\emptyset \in \mathcal{X}$, $X = \bigcup_{\sigma \in \mathcal{X}} \sigma$, and the cells $\sigma \in \mathcal{X}$ are pairwise disjoint;
(2) If $\emptyset \neq \sigma \in \mathcal{X}$, then, for some $i \in \mathbb{N}$, there exists a homeomorphism from an $i$-dimensional ball $\{x \in \mathbb{R}^i \mid ||x|| \leq 1\}$ to the closure $\overline{\sigma}$ of $\sigma$ which maps $\{x \in \mathbb{R}^i \mid ||x|| < 1\}$ onto $\sigma$;
(3) For $\sigma \in \mathcal{X}$, the closure $\overline{\sigma}$ is the union of some cells in $\mathcal{X}$;
(4) For $\sigma, \tau \in \mathcal{X}$, there is a cell $v \in \mathcal{X}$ such that $\overline{v} = \overline{\sigma} \cap \overline{\tau}$ (here $v$ can be $\emptyset$).

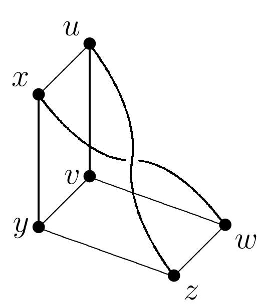
A simplicial complex is a typical example of our $\mathcal{X}$. We regard $\mathcal{X}$ as a partially ordered set (*poset* for short) by $\sigma \geq \tau \overset{\text{def}}{\Longleftrightarrow} \overline{\sigma} \supset \tau$.

**Definition 2.1.** A *conical complex* $(\Sigma, \mathcal{X})$ on $\mathcal{X}$ consists of the following data.

(1) $\Sigma = \{ C_\sigma \mid \sigma \in \mathcal{X} \}$, where $C_\sigma \subset \mathbb{R}^{\dim \sigma + 1}$ is a polyhedral cone with $\dim C_\sigma = \dim \sigma + 1$. (In this paper, "cone" means the one containing no line.)
(2) An injection $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$ satisfying the following.
   (a) $\iota_{\sigma,\tau}$ can be lifted up to a linear map $\tilde{\iota}_{\sigma,\tau} : \mathbb{R}^{\dim \tau + 1} \to \mathbb{R}^{\dim \sigma + 1}$.
   (b) The image $\iota_{\sigma,\tau}(C_\tau)$ is a face of $C_\sigma$. Conversely, for a face $C'$ of $C_\sigma$, there is a *sole* cell $\tau$ with $\tau \leq \sigma$ such that $\iota_{\sigma,\tau}(C_\tau) = C'$.
   (c) $\iota_{\sigma,\sigma} = \mathrm{id}_{C_\sigma}$ and $\iota_{\sigma,\tau} \circ \iota_{\tau,v} = \iota_{\sigma,v}$ for $\sigma, \tau, v \in \mathcal{X}$ with $\sigma \geq \tau \geq v$.

A polyhedral fan in $\mathbb{R}^n$ gives a conical complex. In this case, as an underlying cell complex, we can take $\{$ the relative interior of $C \cap \mathbb{S}^{n-1} \mid C \in \Sigma \}$, where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, and the injections $\iota$ are inclusion maps.

**Example 2.2.** Consider the following cell decomposition of a Möbius strip. Regarding




each rectangles as the cross-sections of 3-dimensional cones, we have a conical complex that is not a fan (see [1]).

Let $\mathbb{R}^{\dim \sigma + 1}$ be the space containing $C_\sigma$, and $\mathbb{Z}^{\dim \sigma + 1} \subset \mathbb{R}^{\dim \sigma + 1}$ the set of lattice points. Assume that $\tilde{\iota}_{\sigma,\tau}(\mathbb{Z}^{\dim \tau + 1}) \subset \mathbb{Z}^{\dim \sigma + 1}$ for all $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$.

**Definition 2.3.** A *monoidal complex* $\mathcal{M}$ supported by a conical complex $(\Sigma, \mathcal{X})$ is a set of monoids $\{\mathbf{M}_\sigma\}_{\sigma \in \mathcal{X}}$ with the following conditions:

(1) $\mathbf{M}_\sigma \subset \mathbb{Z}^{\dim \sigma + 1}$ for each $\sigma \in \mathcal{X}$, and it is a finitely generated additive submonoid (so $\mathbf{M}_\sigma$ is an affine semigroup) with $\mathbb{Z}\mathbf{M}_\sigma = \mathbb{Z}^{\dim \sigma + 1}$;
(2) $\mathbf{M}_\sigma \subset C_\sigma$ and $\mathbb{R}_{\geq 0}\mathbf{M}_\sigma = C_\sigma$ for each $\sigma \in \mathcal{X}$;

(3) for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$, the map $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ induces an isomorphism $\mathbf{M}_\tau \cong \mathbf{M}_\sigma \cap \iota_{\sigma,\tau}(C_\tau)$ of monoids.

For example, let $\Sigma$ be a rational fan in $\mathbb{R}^n$. Then $\{ C \cap \mathbb{Z}^n \mid C \in \Sigma \}$ gives a monoidal complex. More generally, taking a suitable submonoid of $C \cap \mathbb{Z}^n$ for each $C \in \Sigma$, we get a monoidal complex whose monoids are not normal.

For a monoidal complex $\mathcal{M}$, set

$$|\mathcal{M}| := \varinjlim_{\sigma \in \mathcal{X}} \mathbf{M}_\sigma \quad \text{and} \quad |\mathbb{Z}\mathcal{M}| := \varinjlim_{\sigma \in \mathcal{X}} \mathbb{Z}\mathbf{M}_\sigma,$$

where the direct limits are taken with respect to $\iota_{\sigma,\tau} : \mathbf{M}_\tau \to \mathbf{M}_\sigma$ and $\tilde{\iota}_{\sigma,\tau} : \mathbb{Z}\mathbf{M}_\tau \to \mathbb{Z}\mathbf{M}_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$. Note that $|\mathcal{M}|$ (resp. $|\mathbb{Z}\mathcal{M}|$) is just a set and no longer a monoid (resp. abelian group) in general. Since all $\iota_{\sigma,\tau}$ (resp. $\tilde{\iota}_{\sigma,\tau}$) is injective, we can regard $\mathbf{M}_\sigma$ (resp. $\mathbb{Z}\mathbf{M}_\sigma$) as a subset of $|\mathcal{M}|$ (resp. $|\mathbb{Z}\mathcal{M}|$). For example, if $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$, then $|\mathcal{M}| = \bigcup_{\sigma \in \mathcal{X}} \mathbf{M}_\sigma \subset \mathbb{Z}^n$.

Let $a, b \in |\mathbb{Z}\mathcal{M}|$. If there is some $\sigma \in \mathcal{X}$ with $a, b \in \mathbb{Z}\mathbf{M}_\sigma$, by our assumption on $\mathcal{X}$, there is a unique minimal cell among these $\sigma$'s. Hence we can define $a \pm b \in \mathbb{Z}\mathbf{M}_\sigma \subset |\mathbb{Z}\mathcal{M}|$. If there is no $\sigma \in \mathcal{X}$ with $a, b \in \mathbb{Z}\mathbf{M}_\sigma$, then $a \pm b$ do not exist.

**Definition 2.4** ([2])**.** Let $\mathcal{M}$ be a monoidal complex, and $\Bbbk$ a field. Then the $\Bbbk$-vector space

$$\Bbbk[\mathcal{M}] := \bigoplus_{a \in |\mathcal{M}|} \Bbbk\, t^a,$$

where $t$ is a variable, equipped with the following multiplication

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if } a+b \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

has a $\Bbbk$-algebra structure. We call $\Bbbk[\mathcal{M}]$ the *toric face ring* of $\mathcal{M}$ over $\Bbbk$.

Clearly, $\dim \Bbbk[\mathcal{M}] = \dim \mathcal{X} + 1$. In the rest of this paper, we set $d := \dim \Bbbk[\mathcal{M}]$. Stanley-Reisner rings and affine semigroup rings (of positive semigroups) can be established as toric face rings. If $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$, then $\Bbbk[\mathcal{M}]$ admits a $\mathbb{Z}^n$-grading with $\dim_\Bbbk \Bbbk[\mathcal{M}]_a \leq 1$ for all $a \in \mathbb{Z}^n$. But this is not true in general.

**Example 2.5** ([2, Example 4.6])**.** Consider the conical complex given in Example 2.2. Assigning normal semigroup rings of the form $\Bbbk[a,b,c,d]/(ac-bd)$ to the three rectangles, we have a toric face ring of the form

$$\Bbbk[x,y,z,u,v,w]/(xv-uy, vz-yw, xz-uw, uvw, uvz),$$

which does not admit a nice multi-grading. We can also get a similar example whose $\Bbbk[\mathbf{M}_\sigma]$ are not normal.

Let $R := \Bbbk[\mathcal{M}]$ be a toric face ring, and $\operatorname{Mod} R$ the category of $R$-modules.

**Definition 2.6.** $M \in \operatorname{Mod} R$ is said to be $\mathbb{Z}\mathcal{M}$*-graded* if the following are satisfied;

(1) $M = \bigoplus_{a \in |\mathbb{Z}\mathcal{M}|} M_a$ as $\Bbbk$-vector spaces;
(2) Let $a \in |\mathcal{M}|$ and $b \in |\mathbb{Z}\mathcal{M}|$. If $a+b$ exists, then $t^a\, M_b \subset M_{a+b}$. Otherwise, $t^a\, M_b = 0$.

Since $R$ may not be a graded ring in the usual sense, the word "$\mathbb{Z}\mathcal{M}$-graded" is abuse of terminology. An ideal of $R$ is $\mathbb{Z}\mathcal{M}$-graded if and only if it is generated by monomials (i.e., elements of the form $t^a$).

Let $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ denote the subcategory of $\operatorname{Mod} R$ whose objects are $\mathbb{Z}\mathcal{M}$-graded and morphisms are degree preserving (i.e., $f : M \to N$ with $f(M_a) \subset N_a$ for all $a \in |\mathbb{Z}\mathcal{M}|$). It is clear that $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ is an abelian category.

For $\sigma \in \mathcal{X}$, a monomial ideal $\mathfrak{p}_\sigma := (t^a \mid a \in |\mathcal{M}| \setminus \mathbf{M}_\sigma)$ of $R$ is prime. In fact, the quotient ring $\Bbbk[\sigma] := R/\mathfrak{p}_\sigma$ is isomorphic to the affine semigroup ring $\Bbbk[\mathbf{M}_\sigma]$. Conversely, any monomial prime ideal of $R$ is of the form $\mathfrak{p}_\sigma$ for some $\sigma \in \mathcal{X}$.

We say $R$ is *cone-wise normal*, if $\Bbbk[\sigma]$ is normal (equivalently, $\mathbf{M}_\sigma = C_\sigma \cap \mathbb{Z}^{\dim \sigma + 1}$) for all $\sigma \in \mathcal{X}$. Set

$$I_R^{-i} := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \Bbbk[\sigma] = i}} \Bbbk[\sigma]$$

for $i = 0, \dots, d$, and define $\partial : I_R^{-i} \to I_R^{-i+1}$ by

$$\partial(x) = \sum_{\substack{\dim \Bbbk[\tau] = i-1 \\ \tau \le \sigma}} \varepsilon(\sigma, \tau) \cdot f_{\tau,\sigma}(x)$$

for $x \in \Bbbk[\sigma] \subset I_R^{-i}$, where $f_{\tau,\sigma}$ is the natural surjection $\Bbbk[\sigma] \to \Bbbk[\tau]$ (note that if $\tau \le \sigma$ then $\mathfrak{p}_\sigma \subset \mathfrak{p}_\tau$) and $\varepsilon : \mathcal{X} \times \mathcal{X} \to \{0, \pm 1\}$ is an incidence function of $\mathcal{X}$. Then

$$I_R^\bullet : 0 \longrightarrow I_R^{-d} \xrightarrow{\partial} I_R^{-d+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} I_R^0 \longrightarrow 0$$

is a complex. The following is the main result of [6]. Even if $R$ itself is an affine semigroup ring, this does not hold in the non-normal case.

**Theorem 2.7** ([6, Theorem 5.2]). *If $R$ is cone-wise normal, then $I_R^\bullet$ is quasi-isomorphic to the normalized dualizing complex $D_R^\bullet$ of $R$.*

While the word "dualizing complex" sometimes means its isomorphism class in the derived category, we use the convention that a dualizing complex $D_A^\bullet$ of a noetherian ring $A$ is a complex of injective $A$-modules.

For $\sigma \in \mathcal{X}$, set $T_\sigma := \{\, t^a \mid a \in \mathbf{M}_\sigma \,\} \subset R$. Then $T_\sigma$ forms a multiplicatively closed subset consisting of monomials. Well, set

$$L_R^i := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = i-1}} T_\sigma^{-1} R$$

and define $\partial : L_R^i \to L_R^{i+1}$ by

$$\partial(x) = \sum_{\substack{\tau \ge \sigma \\ \dim \tau = i}} \varepsilon(\tau, \sigma) \cdot g_{\tau,\sigma}(x)$$

for $x \in T_\sigma^{-1} R \subset L_R^i$, where $g_{\tau,\sigma}$ is a natural map $T_\sigma^{-1} R \to T_\tau^{-1} R$ for $\sigma \le \tau$. Then $(L_R^\bullet, \partial)$ forms a complex in $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$:

$$L_R^\bullet : 0 \longrightarrow L_R^0 \xrightarrow{\partial} L_R^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_R^d \longrightarrow 0.$$

We set $\mathfrak{m} := (t^a \mid 0 \ne a \in |\mathcal{M}|)$. This is a maximal ideal of $R$.

**Proposition 2.8** (cf. [4, Theorem 4.2])**.** *For $M \in \operatorname{Mod} R$, we have $H^i_{\mathfrak{m}}(M) \cong H^i(M \otimes_R L^\bullet_R)$ for all $i$. In other words, $L^\bullet_R$ is a Cěch complex of $R$ with respect $\mathfrak{m}$.*

The proof for the $\mathbb{Z}^n$-graded case given in [4] also works here. See [6, Proposition 3.2] for detail.

Note that the localization $T_\sigma^{-1}R$ is a $\mathbb{Z}\mathcal{M}$-graded $R$-module, and $L^\bullet_R$ is a $\mathbb{Z}\mathcal{M}$-graded complex. In fact, if we set

$$\mathcal{M} - \mathbf{M}_\sigma := \{\, a - b \mid a \in \mathbf{M}_\tau,\ b \in \mathbf{M}_\sigma,\ \tau \geq \sigma \,\} \subset |\mathbb{Z}\mathcal{M}|$$

Then

$$T_\sigma^{-1}R = \bigoplus_{c \in \mathcal{M} - \mathbf{M}_\sigma} \Bbbk\, t^c.$$

We can define the *($\mathbb{Z}\mathcal{M}$-graded) Matlis duality functor* $(-)^\vee : \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R \to (\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathsf{op}}$ as follows; Let $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$. For $a \in |\mathbb{Z}\mathcal{M}|$ and $b \in |\mathcal{M}|$ such that $a + b$ exists, $(M^\vee)_a$ is the $\Bbbk$-dual space of $M_{-a}$, and the multiplication map $(M^\vee)_a \ni x \mapsto t^b x \in (M^\vee)_{a+b}$ is the $\Bbbk$-dual of $M_{-a-b} \ni y \mapsto t^b y \in M_{-a}$.

In [6, Proposition 5.5], we actually showed the following.

**Proposition 2.9.** *If $R$ is cone-wise normal, then the Matlis dual $(L^\bullet_R)^\vee$ of $L^\bullet_R$ is quasi-isomorphic to the normalized dualizing complex of $R$.*

*Proof.* Under the notation of [6], we have $L^\bullet_R \cong \mathbf{R}\Gamma_{\mathfrak{m}} R$ and $D^\bullet_R = \mathbb{D}(R)$. In the proof of [6, Proposition 5.5], it is shown that $I^\bullet_R$ is a ($\mathbb{Z}\mathcal{M}$-graded) subcomplex of $(L^\bullet_R)^\vee$, and they are quasi-isomorphic. Hence $(L^\bullet_R)^\vee$ is quasi-isomorphic to $D^\bullet_R$ by Theorem 2.7. □

Since $R$ is not a graded ring in the usual sense, the above result is not trivial. The purpose of this paper is to show that it also holds in the non-normal case.

## 3. Main Theorem and Proof

Let the notation be as in the previous section, in particular, $R := \Bbbk[\mathcal{M}]$ is the toric face ring of Krull dimension $d$.

To describe the Matlis dual of the localization $T_\sigma^{-1}R$ for $\sigma \in \mathcal{X}$ explicitly, set

$$\mathbf{M}_\sigma - \mathcal{M} := \{\, a - b \mid a \in \mathbf{M}_\sigma,\ b \in \mathbf{M}_\tau,\ \tau \geq \sigma \,\} \subset |\mathbb{Z}\mathcal{M}|$$

For $c \in \mathbf{M}_\sigma - \mathcal{M}$, let $t^c_\sigma$ be a basis element with degree $c \in |\mathbb{Z}\mathcal{M}|$, and

$$E_\sigma(\mathcal{M}) := \bigoplus_{c \in \mathbf{M}_\sigma - \mathcal{M}} \Bbbk\, t^c_\sigma.$$

Then we can regard $E_\sigma(\mathcal{M})$ as a $\mathbb{Z}\mathcal{M}$-graded $R$-module by

$$t^a \cdot t^c_\sigma = \begin{cases} t^{a+c}_\sigma & \text{if } a, c \in \mathbb{Z}\mathbf{M}_\tau \text{ for some } \tau \geq \sigma \text{ and } a + c \in \mathbf{M}_\sigma - \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $E_\sigma(\mathcal{M})$ is the Matlis dual of the localization $T_\sigma^{-1}R$. As shown in the proof of [4, Theorem 5.1], if $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$ (i.e., $R$ has a nice $\mathbb{Z}^n$-grading), $E_\sigma(\mathcal{M})$ is the injective envelope of $\Bbbk[\sigma]$ in the category of $\mathbb{Z}^n$-graded $R$-modules.

For $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$,

$$t_\sigma^a \mapsto \begin{cases} t_\tau^a & \text{if } a \in \mathbf{M}_\tau - \mathcal{M}, \\ 0 & \text{otherwise} \end{cases}$$

gives an $R$-homomorphism $E_\sigma(\mathcal{M}) \to E_\tau(\mathcal{M})$, which is the Matlis dual $g_{\sigma,\tau}^\vee$ of the natural map $g_{\sigma,\tau} : T_\tau^{-1}R \to T_\sigma^{-1}R$.

Hence the Matlis dual $J_R^\bullet := (L_R^\bullet)^\vee$ of $L_R^\bullet$ has the following form.

$$J_R^\bullet : 0 \to \bigoplus_{\dim \sigma = d-1} E_\sigma(\mathcal{M}) \to \bigoplus_{\dim \sigma = d-2} E_\sigma(\mathcal{M}) \to \cdots \to \bigoplus_{\dim \sigma = 0} E_\sigma(\mathcal{M}) \to E_\emptyset(\mathcal{M}) \to 0.$$

The differentials are give by

$$E_\sigma(\mathcal{M}) \ni x \mapsto \sum_{\substack{\dim \tau = \dim \sigma - 1 \\ \tau \leq \sigma}} \varepsilon(\sigma, \tau) \cdot g_{\sigma,\tau}^\vee(x) \in \bigoplus_{\dim \tau = \dim \sigma - 1} E_\tau(\mathcal{M}),$$

where $\varepsilon$ is the incidence function of $\mathcal{X}$. We put the cohomological degree of $\bigoplus_{\dim \sigma = i-1} E_\sigma(\mathcal{M})$ to $-i$. The following is a main theorem of this paper.

**Theorem 3.1.** *The complex $J_R^\bullet$ is quasi-isomorphic to the normalized dualizing complex $D_R^\bullet$ of $R$.*

*Remark* 3.2. When $R$ itself is an affine semigroup ring, the above theorem was given by Ishida [5] (see also [7]). More precisely, for the semigroup ring $R := \Bbbk[\mathbf{M}]$ of an affine semigroup $\mathbf{M} \subset \mathbb{Z}^n$, $J_R^\bullet$ is a $\mathbb{Z}^n$-graded normalized dualizing complex and quasi-isomorphic to the usual (i.e., non-graded) one. More generally, if $\mathcal{M}$ comes from a fan, then the theorem was given by Ichim and Römer ([4, Theorem 5.1]) by standard argument using the graded ring structure.

For $\sigma, \tau \in \mathcal{X}$ with $\tau \geq \sigma$, let $E_\sigma(\mathbf{M}_\tau)$ be the $\mathbb{Z}\mathcal{M}$-graded Matlis dual of the localization $T_\sigma^{-1}\Bbbk[\tau]$ of $\Bbbk[\tau] = R/\mathfrak{p}_\tau$. Since $\Bbbk[\tau]$ is a quotient of $R$, $E_\sigma(\mathbf{M}_\tau)$ is a submodule of $E_\sigma(\mathcal{M})$ with a $\Bbbk$-basis $\{\, t_\sigma^{a-b} \mid a \in \mathbf{M}_\sigma,\, b \in \mathbf{M}_\tau \,\}$.

By construction, we have $\mathfrak{p}_\tau E_\sigma(\mathbf{M}_\tau) = 0$ and $E_\sigma(\mathbf{M}_\tau)$ can be seen as a $\mathbb{Z}^{\dim \tau + 1}$-graded $\Bbbk[\tau]$-module. In this case, $E_\sigma(\mathbf{M}_\tau)$ is the injective envelope of $\Bbbk[\mathbf{M}_\sigma]$ in the category of $\mathbb{Z}^{\dim \tau + 1}$-graded $\Bbbk[\mathbf{M}_\tau]$-modules.

**Lemma 3.3.** *For $\sigma, \tau \in \mathcal{X}$, we have*

$$\operatorname{Hom}_R(\Bbbk[\tau], E_\sigma(\mathcal{M})) \cong \begin{cases} E_\sigma(\mathbf{M}_\tau) \ (\cong (T_\sigma^{-1}\Bbbk[\tau])^\vee\,) & \text{if } \sigma \leq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Assume that $\sigma \not\leq \tau$. Then we can take $a \in \mathbf{M}_\sigma \backslash \mathbf{M}_\tau$. Clearly, if $b \in \mathbf{M}_\sigma - \mathcal{M}$, then $a + b \in \mathbf{M}_\sigma - \mathcal{M}$, that is, $t^a \cdot t_\sigma^b \neq 0$ in $E_\sigma(\mathcal{M})$. Since $t^a \in \mathfrak{p}_\tau$, we have $\operatorname{Hom}_R(\Bbbk[\tau], E_\sigma(\mathcal{M})) = 0$.

Next, assume that $\sigma \leq \tau$. The inclusion

$$\operatorname{Hom}_R(\Bbbk[\tau], E_\sigma(\mathcal{M})) \cong \{\, x \in E_\sigma(\mathcal{M}) \mid \mathfrak{p}_\tau x = 0 \,\} \supset E_\sigma(\mathbf{M}_\tau)$$

is clear. To show the opposite inclusion, take $x \in E_\sigma(\mathcal{M}) \backslash E_\sigma(\mathbf{M}_\tau)$. We may assume that $x = t_\sigma^{a-b}$ for some $a \in \mathbf{M}_\sigma$ and $b \notin \mathbf{M}_\tau$. Then $t^b \in \mathfrak{p}_\tau$ and $t^b \cdot t_\sigma^{a-b} = t_\sigma^a \neq 0$. $\square$

The next result easily follows from the above discussion (and Remark 3.2).

**Lemma 3.4.** *For $\sigma \in \mathcal{X}$, the complex $\operatorname{Hom}^\bullet_R(\Bbbk[\sigma], J^\bullet_R)$ is isomorphic to $J^\bullet_{\Bbbk[\sigma]}$, and quasi-isomorphic to the dualizing complex $D^\bullet_{\Bbbk[\sigma]} = \operatorname{Hom}^\bullet_R(\Bbbk[\sigma], D^\bullet_R)$ of $\Bbbk[\sigma]$.*

For each $\sigma \in \mathcal{X}$, set $\overline{\mathbf{M}}_\sigma := \mathbb{Z}^{\dim \sigma+1} \cap C_\sigma$. Then $\overline{\mathcal{M}} := \{\, \overline{\mathbf{M}}_\sigma \,\}_{\sigma \in \mathcal{X}}$ is a monoidal complex supported by $\mathcal{X}$ again. Let $\widetilde{R} := \Bbbk[\overline{\mathcal{M}}]$ be the toric face ring of $\overline{\mathcal{M}}$. For the monomial prime ideal $\widetilde{\mathfrak{p}}_\sigma$ of $\widetilde{R}$ associated with $\sigma \in \mathcal{X}$, we have $\widetilde{R}/\widetilde{\mathfrak{p}}_\sigma \cong \Bbbk[\overline{\mathbf{M}}_\sigma]$ and this is the normalization $\overline{\Bbbk[\sigma]}$ of $\Bbbk[\sigma]$. So we denote $\widetilde{R}/\widetilde{\mathfrak{p}}_\sigma$ by $\overline{\Bbbk[\sigma]}$. Since $\widetilde{R}$ is cone-wise normal, $J^\bullet_{\widetilde{R}}$ is quasi-isomorphic to $D^\bullet_{\widetilde{R}}$ by Theorem 2.7. Moreover, we have the following.

**Lemma 3.5.** *There is a quasi-isomorphism $\psi : J^\bullet_{\widetilde{R}} \to D^\bullet_{\widetilde{R}}$ such that the induced map $\psi_\sigma := \operatorname{Hom}^\bullet_{\widetilde{R}}(\overline{\Bbbk[\sigma]}, \psi) : J^\bullet_{\overline{\Bbbk[\sigma]}} \to D^\bullet_{\overline{\Bbbk[\sigma]}}$ is a quasi-isomorphism for all $\sigma \in \mathcal{X}$.*

*Proof.* In [6], we showed that $I^\bullet_{\widetilde{R}}$ can be seen as a subcomplex of $D^\bullet_{\widetilde{R}}$. This gives a quasi-isomorphism $\eta : I^\bullet_{\widetilde{R}} \to D^\bullet_{\widetilde{R}}$ such that the induced map $\eta_\sigma := \operatorname{Hom}^\bullet_{\widetilde{R}}(\overline{\Bbbk[\sigma]}, \eta) : I^\bullet_{\overline{\Bbbk[\sigma]}} \to D^\bullet_{\overline{\Bbbk[\sigma]}}$ is a quasi-isomorphism again for all $\sigma \in \mathcal{X}$.

Since $\widetilde{R}$ is cone-wise normal, $I^\bullet_{\widetilde{R}}$ is a $\mathbb{Z}\mathcal{M}$-graded subcomplex of $J^\bullet_{\widetilde{R}}$, and the chain map $\iota : I^\bullet_{\widetilde{R}} \hookrightarrow J^\bullet_{\widetilde{R}}$ is a quasi-isomorphism as pointed out in the proof of Proposition 2.9. The diagram $J^\bullet_{\widetilde{R}} \xleftarrow{\ \iota\ } I^\bullet_{\widetilde{R}} \xrightarrow{\ \eta\ } D^\bullet_{\widetilde{R}}$ gives an isomorphism $J^\bullet_{\widetilde{R}} \to D^\bullet_{\widetilde{R}}$ in $\mathsf{D}^b(\operatorname{Mod} R)$. Since $D^\bullet_{\widetilde{R}}$ is a complex of injective $R$-modules, there is an actual chain map $\psi : J^\bullet_{\widetilde{R}} \to D^\bullet_{\widetilde{R}}$ giving this isomorphism. Clearly, $\psi$ is a quasi-isomorphism and $\eta$ is homotopic to $\psi \circ \iota$. It is easy to see that $\iota_\sigma := \operatorname{Hom}^\bullet_{\widetilde{R}}(\overline{\Bbbk[\sigma]}, \iota) : I^\bullet_{\overline{\Bbbk[\sigma]}} \to J^\bullet_{\overline{\Bbbk[\sigma]}}$ is a quasi-isomorphism, and $\eta_\sigma$ is homotopic to $\psi_\sigma \circ \iota_\sigma$. Since $\eta_\sigma$ and $\iota_\sigma$ are quasi-isomorphisms, so is $\psi_\sigma$. □

Since $\widetilde{R}$ is finitely generated as an $R$-module, we have $D^\bullet_{\widetilde{R}} = \operatorname{Hom}^\bullet_R(\widetilde{R}, D^\bullet_R)$. Via the canonical injection $R \hookrightarrow \widetilde{R}$, we have a chain map $\lambda : D^\bullet_{\widetilde{R}} = \operatorname{Hom}^\bullet_R(\widetilde{R}, D^\bullet_R) \longrightarrow \operatorname{Hom}^\bullet_R(R, D^\bullet_R) = D^\bullet_R$. Similarly, for each $\sigma \in \mathcal{X}$, the injection $\Bbbk[\sigma] \hookrightarrow \overline{\Bbbk[\sigma]}$ induces a chain map $\lambda_\sigma : D^\bullet_{\overline{\Bbbk[\sigma]}} \to D^\bullet_{\Bbbk[\sigma]}$.

As a $\mathbb{Z}^{\dim \sigma+1}$-graded version of the well-known fact $D^\bullet_{\overline{\Bbbk[\sigma]}} = \operatorname{Hom}^\bullet_{\Bbbk[\sigma]}(\overline{\Bbbk[\sigma]}, D^\bullet_{\Bbbk[\sigma]})$, we have $J^\bullet_{\overline{\Bbbk[\sigma]}} = \operatorname{Hom}^\bullet_{\Bbbk[\sigma]}(\overline{\Bbbk[\sigma]}, J^\bullet_{\Bbbk[\sigma]})$. Similarly, we have a chain map $\mu_\sigma : J^\bullet_{\overline{\Bbbk[\sigma]}} \to J^\bullet_{\Bbbk[\sigma]}$ which is the $\mathbb{Z}^{\dim \sigma+1}$-graded version of $\lambda_\sigma$.

**Lemma 3.6.** *For the quasi-isomorphism $\psi_\sigma : J^\bullet_{\overline{\Bbbk[\sigma]}} \to D^\bullet_{\overline{\Bbbk[\sigma]}}$, we have a quasi-isomorphism $\phi_\sigma : J^\bullet_{\Bbbk[\sigma]} \to D^\bullet_{\Bbbk[\sigma]}$ which makes the following diagram commutative.*

$$\begin{tikzcd}
J^\bullet_{\overline{\Bbbk[\sigma]}} \arrow[r, "\psi_\sigma"] \arrow[d, "\mu_\sigma"'] & D^\bullet_{\overline{\Bbbk[\sigma]}} \arrow[d, "\lambda_\sigma"] \\
J^\bullet_{\Bbbk[\sigma]} \arrow[r, "\phi_\sigma"'] & D^\bullet_{\Bbbk[\sigma]}
\end{tikzcd}$$

*Proof.* Let $\xi : J^\bullet_{\Bbbk[\sigma]} \to D^\bullet_{\Bbbk[\sigma]}$ be an arbitrary quasi-isomorphism. Since $\overline{\Bbbk[\sigma]}$ is a $\mathbb{Z}^{\dim\sigma+1}$-graded $\Bbbk[\sigma]$-module and $J^\bullet_{\Bbbk[\sigma]}$ is a complex of $\mathbb{Z}^{\dim\sigma+1}$-graded injective $\Bbbk[\sigma]$-modules, $\xi$ gives a quasi-isomorphism

$$\xi_* : J^\bullet_{\overline{\Bbbk[\sigma]}} = \mathrm{Hom}^\bullet_{\Bbbk[\sigma]}(\overline{\Bbbk[\sigma]}, J^\bullet_{\Bbbk[\sigma]}) \longrightarrow \mathrm{Hom}^\bullet_{\Bbbk[\sigma]}(\overline{\Bbbk[\sigma]}, D^\bullet_{\Bbbk[\sigma]}) = D^\bullet_{\overline{\Bbbk[\sigma]}}.$$

Since, as is well-known, $\mathrm{Hom}_{\mathsf{D}^b(\mathrm{Mod}\,\overline{\Bbbk[\sigma]})}(D^\bullet_{\overline{\Bbbk[\sigma]}}, D^\bullet_{\overline{\Bbbk[\sigma]}}) = \overline{\Bbbk[\sigma]}$, we have $\psi_\sigma = c\,\xi_*$ for some $0 \neq c \in \Bbbk$. Hence $\phi_\sigma := c\,\xi$ satisfies the expected condition. □

For each $i \in \mathbb{Z}$, $J^i_R$ is a $\mathbb{Z}\mathcal{M}$-graded submodule of $J^i_{\widetilde{R}}$ (here we regard $J^i_{\widetilde{R}}$ as an $R$-module), moreover, $J^i_R$ is a direct summand of $J^i_{\widetilde{R}}$. However $J^\bullet_R$ is NOT a subcomplex of $J^\bullet_{\widetilde{R}}$. Let $\kappa : J^\bullet_R \dashrightarrow J^\bullet_{\widetilde{R}}$ be the component-wise injection (since this is not a chain map, we use the symbol "$\dashrightarrow$"). For the similar map $\kappa_\sigma : J^\bullet_{\Bbbk[\sigma]} \dashrightarrow J^\bullet_{\overline{\Bbbk[\sigma]}}$, we have $\mu^i_\sigma \circ \kappa^i_\sigma = \mathrm{Id}$ for all $i$.

**Lemma 3.7.** *The composition of*

$$J^\bullet_R \overset{\kappa}{\dashrightarrow} J^\bullet_{\widetilde{R}} \xrightarrow{\psi} D^\bullet_{\widetilde{R}} \xrightarrow{\lambda} D^\bullet_R$$

*is a chain map.*

*Proof.* For any $x \in J^i_R$, there is some $\sigma \in \mathcal{X}$ such that $\mathfrak{p}_\sigma x = 0$. Regarding $J^\bullet_{\Bbbk[\sigma]} = \mathrm{Hom}^\bullet_R(\Bbbk[\sigma], J^\bullet_R)$ as a subcomplex of $J^\bullet_R$, we have $x \in J^i_{\Bbbk[\sigma]}$. Since $J^\bullet_{\overline{\Bbbk[\sigma]}}$ (resp. $D^\bullet_{\overline{\Bbbk[\sigma]}}$ and $D^\bullet_{\Bbbk[\sigma]}$) can be seen as a subcomplex of $J^i_{\widetilde{R}}$ (resp. $D^\bullet_{\widetilde{R}}$ and $D^\bullet_R$), we have the following commutative diagram.

$$\begin{tikzcd}
J^i_{\Bbbk[\sigma]} \arrow[r, "\kappa_\sigma"] \arrow[d, hook] & J^i_{\overline{\Bbbk[\sigma]}} \arrow[r, "\psi_\sigma"] \arrow[d, hook] & D^i_{\overline{\Bbbk[\sigma]}} \arrow[r, "\lambda_\sigma"] \arrow[d, hook] & D^i_{\Bbbk[\sigma]} \arrow[d, hook] \\
J^i_R \arrow[r, "\kappa"'] & J^i_{\widetilde{R}} \arrow[r, "\psi"'] & D^i_{\widetilde{R}} \arrow[r, "\lambda"'] & D^i_R
\end{tikzcd}$$

By Lemma 3.6, we have $\lambda^i_\sigma \circ \psi^i_\sigma \circ \kappa^i_\sigma = \phi^i_\sigma \circ \mu^i_\sigma \circ \kappa^i_\sigma = \phi^i_\sigma$. Since $\phi_\sigma$ is a chain map, we are done. □

We denote the chain map $J^\bullet_R \to D^\bullet_R$ constructed in Lemma 3.7 by $\phi$. To prove Theorem 3.1, we will show that $\phi$ is a quasi-isomorphism by a slightly indirect way.

For each $a \in |\mathcal{M}|$, there is a unique minimal element among the cells $\sigma \in \mathcal{X}$ such that $a \in \mathbf{M}_\sigma$. We denote this minimal cell by $\mathrm{supp}(a)$.

**Definition 3.8.** An $R$-module $M \in \mathrm{Mod}_{\mathbb{Z}\mathcal{M}} R$ is said to be *squarefree* if it is $\mathcal{M}$-graded (i.e., $M = \bigoplus_{a\in|\mathcal{M}|} M_a$), finitely generated, and the multiplication map $M_a \ni x \mapsto t^b x \in M_{a+b}$ is bijective for all $a, b \in |\mathcal{M}|$ such that $a + b$ exists and $\mathrm{supp}(a + b) = \mathrm{supp}(a)$.

The notion of squarefree modules over a normal semigroup ring was introduced by the author ([9]), and many applications have been found. In [6], squarefree modules over a cone-wise normal toric face ring play a key role. Contrary to the (cone-wise) normal case, the derived category of squarefree modules is *not* closed

under the duality $\mathbf{R}\operatorname{Hom}_R(-, D_R^\bullet) = \operatorname{Hom}_R^\bullet(-, D_R^\bullet)$. However, these modules still enjoy some nice properties.

**Lemma 3.9** (cf. [6, Lemma 4.2]). *Let $M \in \operatorname{Sq} R$. Then for $a, b \in |\mathcal{M}|$ with $\operatorname{supp}(a) \geq \operatorname{supp}(b)$, there exists a $\Bbbk$-linear map $\varphi_{a,b}^M : M_b \to M_a$ satisfying the following properties:*

(1) *If $a = b + c$, then $\varphi_{a,b}^M$ coincides with the multiplication map $M_b \ni x \mapsto t^c x \in M_a$;*
(2) *$\varphi_{a,b}^M$ is bijective if $\operatorname{supp}(a) = \operatorname{supp}(b)$;*
(3) *$\varphi_{a,b}^M \circ \varphi_{b,c}^M = \varphi_{a,c}^M$ for $a, b, c \in |\mathcal{M}|$ with $\operatorname{supp}(c) \leq \operatorname{supp}(b) \leq \operatorname{supp}(a)$.*

*Proof.* The proof for the normal case works here, but we repeat it for the reader's convenience. Set $\varphi_{a+b,b}^M : M_b \to M_{a+b}$ to be the multiplication map $M_b \ni x \mapsto t^a x \in M_{a+b}$, and define $\varphi_{a+b,a}^M : M_a \to M_{a+b}$ in the same way. Since $\operatorname{supp}(a) = \operatorname{supp}(a+b)$, $\varphi_{a+b,a}^M$ is bijective and we can put $\varphi_{a,b}^M := (\varphi_{a+b,a}^M)^{-1} \circ \varphi_{a+b,b}^M$. □

Let $\operatorname{Sq} R$ be the full subcategory of $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ consisting of squarefree modules. By virtue of the above lemma, [6, Lemma 4.2] remains true in the present case.

**Lemma 3.10** (c.f. [6, Lemma 4.2]). *The category $\operatorname{Sq} R$ is equivalent to the category of finitely generated left $\Lambda$-modules, where $\Lambda$ is the incidence algebra of the poset $\mathcal{X}$ over $\Bbbk$. Hence $\operatorname{Sq} R$ is an abelian category with enough injectives, and indecomposable injectives are objects isomorphic to $\Bbbk[\sigma]$ for some $\sigma \in \mathcal{X}$. The injective dimension of any object is at most $d$.*

Recall that if $M$ is a $\mathbb{Z}\mathcal{M}$-graded $R$-module, then the localization $T_\sigma^{-1} M$ is also. Since $L_R^\bullet$ is a complex of flat $R$-modules, $(- \otimes_R L_R^\bullet)$ gives an exact functor $\mathsf{D}^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R) \to \mathsf{D}^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)$. Composing this one and the Matlis duality, we have an exact functor $(- \otimes_R L_R^\bullet)^\vee : \mathsf{D}^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R) \to \mathsf{D}^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathsf{op}}$.

Let Inj-Sq be the full subcategory of $\operatorname{Sq} R$ consisting of all injective objects, that is, finite direct sums of $\Bbbk[\sigma]$ for various $\sigma \in \mathcal{X}$. As is well-known (cf. [3, Proposition I.4.7]), the bounded homotopy category $\mathsf{K}^b(\text{Inj-Sq})$ is equivalent to $\mathsf{D}^b(\operatorname{Sq} R)$. It is easy to see that the functor $(- \otimes_R L_R^\bullet)^\vee : \mathsf{D}^b(\operatorname{Sq} R) \to \mathsf{D}^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathsf{op}}$ can be identified with $\operatorname{Hom}_R^\bullet(-, J_R^\bullet) : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{D}^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathsf{op}}$ by Lemma 3.3. Via the forgetful functor $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R \to \operatorname{Mod} R$, we get an exact functor

$$\operatorname{Hom}_R^\bullet(-, J_R^\bullet) : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{D}^b(\operatorname{Mod} R)^{\mathsf{op}}.$$

Since $D_R^\bullet$ is a complex of injective $R$-modules, $\operatorname{Hom}_R^\bullet(-, D_R^\bullet)$ gives an exact functor $\mathsf{D}^b(\operatorname{Mod} R) \to \mathsf{D}^b(\operatorname{Mod} R)^{\mathsf{op}}$. Similarly, we have an exact functor

$$\operatorname{Hom}_R^\bullet(-, D_R^\bullet) : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{D}^b(\operatorname{Mod} R)^{\mathsf{op}}.$$

The chain map $\phi : J_R^\bullet \to D_R^\bullet$ gives a natural transformation

$$\Phi : \operatorname{Hom}_R^\bullet(-, J_R^\bullet) \to \operatorname{Hom}_R^\bullet(-, D_R^\bullet).$$

**Theorem 3.11.** *The natural transformation $\Phi$ is an natural isomorphism.*

*Proof.* By virtue of [3, Proposition 7.1], it suffices to show that $\Phi(\Bbbk[\sigma]) : J^\bullet_{\Bbbk[\sigma]} = \operatorname{Hom}^\bullet_R(\Bbbk[\sigma], J^\bullet_R) \to \operatorname{Hom}^\bullet_R(\Bbbk[\sigma], D^\bullet_R) = D^\bullet_{\Bbbk[\sigma]}$ is quasi-isomorphism for all $\sigma \in \mathcal{X}$. Since $\Phi(\Bbbk[\sigma]) = \operatorname{Hom}^\bullet_R(\Bbbk[\sigma], \phi)$, it is factored as

$$J^\bullet_{\Bbbk[\sigma]} \overset{\kappa_\sigma}{\dashrightarrow} J^\bullet_{\overline{\Bbbk[\sigma]}} \xrightarrow{\psi_\sigma} D^\bullet_{\overline{\Bbbk[\sigma]}} \xrightarrow{\lambda_\sigma} D^\bullet_{\Bbbk[\sigma]},$$

while $\kappa_\sigma$ is just a "component-wise map". As shown in the proof of Lemma 3.7, this coincides with the quasi-isomorphism $\phi_\sigma$ of Lemma 3.6. $\square$

*The proof of Theorem 3.1.* The theorem follows from Theorem 3.11. In fact, $R \in \operatorname{Sq} R$ and $\phi : J^\bullet_R \to D^\bullet_R$ coincides with $\Phi(R) : \operatorname{Hom}^\bullet_R(R, J^\bullet_R) \to \operatorname{Hom}^\bullet_R(R, D^\bullet_R)$. $\square$

**Corollary 3.12.** *$R$ is Cohen-Macaulay if and only if so is the local ring $R_\mathfrak{m}$.*

*Proof.* By Theorem 3.1, $R$ is Cohen-Macaulay if and only if $H^i(J^\bullet_R) = 0$ for all $i \neq -d$. Since $H^i(J^\bullet_R)$ is $\mathbb{Z}\mathcal{M}$-graded, $H^i(J^\bullet_R) \neq 0$ implies $\mathfrak{m} \in \operatorname{Supp}(H^i(J^\bullet_R))$ by Lemma 3.13 below. Hence $R$ is Cohen-Macaulay, if and only if $H^i(J^\bullet_R) \otimes R_\mathfrak{m} = 0$ for all $i \neq -d$, if and only if $R_\mathfrak{m}$ is Cohen-Macaulay. $\square$

**Lemma 3.13.** *If $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ is finitely generated, then any associated prime of $M$ is of the form $\mathfrak{p}_\sigma$ for some $\sigma \in \mathcal{X}$.*

*Proof.* Let $\mathfrak{p}$ be an associated prime of $M$. Since any minimal prime of $R$ is of the form $\mathfrak{p}_\tau$ for a maximal cell $\tau \in \mathcal{X}$ (see [6]), there is some $\tau \in \mathcal{X}$ with $\mathfrak{p}_\tau \subset \mathfrak{p}$. The submodule $M' := \{\, y \in M \mid \mathfrak{p}_\tau y = 0 \,\}$ of $M$ is a $\mathbb{Z}^{\dim \tau + 1}$-graded $\Bbbk[\tau]$-module, and the image $\bar{\mathfrak{p}}$ of $\mathfrak{p}$ in $\Bbbk[\tau]$ is an associated prime of $M'$. Hence $\mathfrak{p}$ is $\mathbb{Z}^{\dim \tau + 1}$-graded, and $\mathfrak{p} = \mathfrak{p}_\sigma$ for some $\sigma \in \mathcal{X}$ with $\sigma \leq \tau$. $\square$

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